End of proof of the subadditive ergodic theorem and applications

Thursday, 29 April 2021 13:07

$X_{0,1}$
$X_{0,2}, X_{1,2}$
$X_{0,3}$

Example: $\{\xi_m\}_{m\ge 1}$ stationary.
$X_{m,n} := \xi_{m+1} + \cdots + \xi_n$. $\leftarrow X_{k,r} = X_{k,\ell} + X_{\ell,r}$. (additivity)

The theorem (Liggett's version (1985), improve from Kingman's version (1968)):

Suppose $(X_{m,n})$, $0 \le m < n$, satisfy

i) $X_{0,n} \le X_{0,m} + X_{m,n}$ for $0 \le m < n$. (subadditivity)

ii) For each $i \ge 1$,
$(X_{0,i}, X_{i,2i}, X_{2i,3i}, \cdots)$ is stationary. (generalized diagonals of the triangular array are stationary)

iii) For each $m \ge 1$,
$(X_{0,1}, X_{0,2}, X_{0,3}, \cdots) \overset{d}{=} (X_{m,m+1}, X_{m,m+2}, X_{m,m+3}, \cdots)$. (columns of the triangular array have the same dist.)

iv) $\mathbb{E}\underbrace{X_{0,1}^+}_{\max\{X_{0,1},0\}} < \infty$ and for each n, $\mathbb{E}X_{0,n} \ge \gamma_0 n$ for some $\gamma_0 > -\infty$. (moment condition)

proved last time

Then (a) $\lim_{n\to\infty} \frac{\mathbb{E}X_{0,n}}{n} = \inf_n \frac{\mathbb{E}X_{0,n}}{n}$ exists. Denote it by γ.

(b) $X := \lim_{n\to\infty} \frac{1}{n} X_{0,n}$ exists a.s. and in L^1. In particular, $\mathbb{E}X = \gamma$.

(c) If the stationary seq. in (ii) are ergodic then $X = \gamma$ a.s.

Proved (a) last time (step 1).
Also, in step 2, we defined
$$\overline{X} := \limsup_{n\to\infty} \frac{X_{0,n}}{n}$$
and proved $\mathbb{E}\overline{X} \le \gamma$.

Continuation of proof: Step 3: $\underline{X} := \liminf_{n\to\infty} \frac{X_{0,n}}{n}$.

Continuation of proof: Step 3: $\underline{X} := \liminf_{n\to\infty} \frac{X_{0,n}}{n}$.

In this step we prove that $\overline{X} = \underline{X}$ a.s. and the limit of $\frac{X_{0,n}}{n}$ exists a.s.

To this end, we show that $\mathbb{E}\underline{X} \geq \gamma$.

Since $\underline{X} \leq \overline{X}$ and $\mathbb{E}\overline{X} \leq \gamma$ (Step 2)

So this implies that $\underline{X} = \overline{X}$ a.s.

(Using also that γ is finite, $\gamma < \infty$ (Step 1)

Additionally, $\mathbb{E}X_{0,n} \geq \gamma_0 n$ for $\gamma_0 > -\infty$ (iv)

and hence $\gamma = \inf \frac{\mathbb{E}X_{0,n}}{n} \geq \gamma_0 > -\infty$).

Idea: Truncate $\underline{X}$ to $\underbrace{\underline{X} \vee (-M)}_{\max\{\underline{X}, -M\}}$ and define a new triangular array by subtracting $\underline{X} \vee (-M)$. Then we will give an upper bound to a limit of the new triangular array, and thereby lower bound $\underline{X} \vee (-M)$.

Fix $\varepsilon > 0$ and $M > 0$.

Define $Z = \varepsilon + (\underline{X} \vee (-M))$.

Note that $\mathbb{E}|Z| < \infty$ since Z is bounded from below and $\underline{X} \leq \overline{X}$ and $\mathbb{E}\overline{X} \leq \gamma$.

New array: $Y_{m,n} := X_{m,n} - (n-m)Z$.

Assumptions (i)-(iv) hold for $(Y_{m,n})$:

(i) Subadditivity: $Y_{0,n} \leq Y_{0,m} + Y_{m,n}$, OK.

(ii) For each i, $(Y_{0,i}, Y_{i,2i}, \ldots)$ is stationary.

(ii) For each j, $(Y_{0,j}, Y_{j,2j}, \ldots)$ is stationary.

To this end we investigate $\underline{X}$ a bit further.

Define $\underline{X}_m := \liminf_{n\to\infty} \frac{1}{n} X_{m,m+n}$,

so that $\underline{X} = \underline{X}_0$.

Claim: $\underline{X}_m = \underline{X}$ for all m, a.s.

Proof: By (iii), $\underline{X}_m \overset{d}{=} \underline{X}$ for all m.

By the subadditivity (i),

$$\underline{X} = \liminf_{n\to\infty} \frac{1}{m+n} X_{0,m+n} \le \liminf_{n\to\infty} \frac{1}{m+n}(X_{0,m} + X_{m,m+n})$$

$$= 0 + \underline{X}_m$$

Now $\underline{X} \le \underline{X}_m$ and $\underline{X} \overset{d}{=} \underline{X}_m$ implies $\underline{X} = \underline{X}_m$ a.s..

The claim suffices to deduce that $(Y_{m,n})$ satisfies (ii)

iii) For each m, $(Y_{0,1}, Y_{0,2}, \ldots) \overset{d}{=} (Y_{m,m+1}, Y_{m,m+2}, \ldots)$

again follows from (iii) for $(X_{m,n})$ and the claim.

iv) Moment cond. follow from $\mathbb{E}|Z| < \infty$ and the moment cond. on X.

We proceed to upper bound $\limsup_{n\to\infty} \frac{\mathbb{E}\, Y_{0,n}}{n}$,

and this will give a lower bound for $\mathbb{E}Z$.

First, note that $\liminf_{n\to\infty} \frac{Y_{0,n}}{n} \le -\varepsilon$, by def. of $(Y_{m,n})$.

Idea: Choose a seq. of times $0 = R_0 < R_1 < \ldots < R_K \le n$

and write $Y_{0,n} \le Y_{0,R_1} + Y_{R_1,R_2} + \ldots + Y_{R_K,n}$

using subadditivity. The seq. of times

will be chosen so that $Y_{R_i,R_{i+1}}$ is small

and we will need to control the remainder

will be chosen ... R_i, R_{i+1} is
and we will need to control the remainder $Y_{R_K, n}$.

Let $T_m = \min\{n \geq 1 : Y_{m,m+n} \leq 0\}$.

We have that $T_m < \infty$ a.s. since $\liminf_{n\to\infty} \frac{Y_{m,m+n}}{n} \leq -\varepsilon$

Let N be an integer, chosen large enough for the following.

Define $$S_m := \begin{cases} T_m & T_m \leq N \\ 1 & T_m > N \end{cases}$$

$R_0 := 0$, $R_i := R_{i-1} + S_{R_{i-1}}$ for $i \geq 1$.

Then, by subadditivity,

$$Y_{0,n} \leq Y_{R_0,R_1} + Y_{R_1,R_2} + \dots + Y_{R_{K-1},R_K} + Y_{R_K,n}$$

where $K = \max\{k : R_k \leq n\}$.

In particular, $n - R_K \leq N$.

Define $$\xi_m := \begin{cases} 0 & T_m \leq N \\ Y_{m,m+1} & T_m > N \end{cases}.$$

With this definition, $Y_{R_i,R_{i+1}} \leq \xi_{R_i}$.

Note also that $\xi_m \geq 0$ for all m.

We conclude that

$$Y_{R_0,R_1} + \dots + Y_{R_{K-1},R_K} + Y_{R_K,n} \leq$$ by subadditivity applied to the last term

$$\sum_{i=0}^{n-1} \xi_i + \sum_{j=1}^{N} |Y_{n-j,n-j+1}|$$ Note that this bound doesn't depend ...

$\sum_{i=0} \xi_i + \sum_{j=1} |Y_{n-j,n-j+1}|$ ← Note that this bound doesn't depend on the R's.

Consequently,

$$\frac{1}{n}\mathbb{E}Y_{0,n} \le \frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}\xi_i + \underbrace{\frac{1}{n}\sum_{j=1}^{N}\mathbb{E}|Y_{n-j,n-j+1}|}_{\overset{\text{Stationarity (iii)}}{=} \frac{1}{n}\sum_{j=1}^{N}\mathbb{E}|Y_{j-1,j}| \xrightarrow[n\to\infty]{} 0}$$

For the first sum,

$$\mathbb{E}\xi_m = \mathbb{E}(Y_{m,m+1}\cdot 1_{T_m > N}) \overset{\text{Stationarity (iii)}}{=} \mathbb{E}(Y_{0,1}\cdot 1_{T_0>N}) \xrightarrow[N\to\infty]{} 0$$

by the dominated convergence thm., since $T_0 < \infty$ a.s. and $\mathbb{E}|Y_{0,1}| < \infty$.

Fix an N s.t. $\mathbb{E}\xi_m \le \varepsilon$. Thus, finally,

$$\limsup_{n\to\infty} \frac{1}{n}\mathbb{E}(Y_{0,n}) \le \varepsilon.$$

Recall, $Y_{0,n} = X_{0,n} - nZ$ and $Z = \varepsilon + (\underline{X} \vee (-M))$.

Since $\lim_{n\to\infty} \frac{1}{n}\mathbb{E}X_{0,n} = \gamma$,

$$\gamma - \mathbb{E}Z \le \varepsilon \Rightarrow \mathbb{E}(\underline{X} \vee (-M)) \ge \gamma - 2\varepsilon.$$

Since ε, M are arbitrary, get $\mathbb{E}\underline{X} \ge \gamma$ as we wanted to show.

Conclusion: $\lim_{n\to\infty}\frac{1}{n}X_{0,n}$ exists a.s.

Step 4: Here we show that $\lim_{n\to\infty}\frac{1}{n}X_{0,n}$ exists in L_1.

Reminder from Step 1: For each m, we considered

$$\frac{X_{0,m} + X_{m,2m} + \ldots + X_{(k-1)m,km}}{\ldots} \to A.$$

considered $\dfrac{X_{0,m}+X_{m,2m}+\dots+X_{(k-1)m,km}}{k} \xrightarrow[k\to\infty]{} A_m$ a.s. and in L_1

by the ergodic theorem,

$A_m = \mathbb{E}(X_{0,m} \mid \mathcal{I}_m)$ invariant σ-alg. of $(X_{0,m}, X_{m,2m}, \dots)$

We also showed that

$$\overline{X} = \limsup_{n\to\infty} \frac{1}{n} X_{0,n} \le \frac{1}{m} A_m \text{ for each } m.$$

Lastly, $\mathbb{E}(A_m) = \mathbb{E}(X_{0,m}) \Rightarrow \frac{1}{m}\mathbb{E}A_m \xrightarrow[m\to\infty]{} \gamma = \inf_m \frac{\mathbb{E}A_m}{m}$.

Now define $T_m := \frac{1}{m} A_m$ and $T := \inf_m T_m$.

We will prove that $\frac{1}{n} X_{0,n} \to T$ in L_1.

$$\mathbb{E}\left|\frac{1}{n}X_{0,n} - T\right| \le 2\mathbb{E}\left[\left(\frac{1}{n}X_{0,n} - T\right)^+\right] - \underbrace{\mathbb{E}\left(\frac{1}{n}X_{0,n} - T\right)}_{=\frac{1}{n}\mathbb{E}A_n - \mathbb{E}\left(\inf_n \frac{A_n}{n}\right) \ge 0}$$

$|x| = 2x^+ - x,\ x \in \mathbb{R}$

$$\le 2\mathbb{E}\left[\left(\frac{1}{n}X_{0,n} - T\right)^+\right] \le$$

$\leftarrow$ for any m, $(x+y)^+ \le x^+ + y^+$ for $x, y \in \mathbb{R}$

$$\le 2\cdot\left(\mathbb{E}\left[\left(\frac{1}{n}X_{0,n} - T_m\right)^+\right] + \mathbb{E}\left[\left(T_m - T\right)^+\right]\right)$$

For second factor, Claim: $\mathbb{E}T = \gamma$

Proof: By def., $\gamma = \inf_m \mathbb{E}T_m$. Hence $\mathbb{E}T = \mathbb{E}\inf_m T_m \le \gamma$.

Additionally, $\mathbb{E}\overline{X} \le \mathbb{E}T_m \le \mathbb{E}T$

and we saw in step 3 that $\mathbb{E}\overline{X} \ge \mathbb{E}\underline{X} \ge \gamma$.

By the claim, and the fact that $T_m \le T$ by def., By the claim, and the fact that $T_m \geq T$ by def.,

$$\mathbb{E}\left((T_m - T)^+\right) = \mathbb{E}(T_m - T) \xrightarrow[m\to\infty]{} 0.$$

For the first term, write $n = km + \ell$, $0 \leq \ell < m$,

$$\mathbb{E}\left[\left(\frac{1}{n}X_{0,n} - T_m\right)^+\right] \overset{\text{subadditivity}}{\leq} \mathbb{E}\left[\left(\frac{1}{n}(X_{0,m} + X_{m,2m} + \dots + X_{(k-1)m,km}) + \frac{1}{n}X_{km,n} - T_m\right)^+\right] \leq$$

$$\leq \underbrace{\mathbb{E}\left|\frac{1}{km+\ell}(X_{0,m} + \dots + X_{(k-1)m,km}) - T_m\right|}_{\xrightarrow[k\to\infty]{} 0,\ \text{by the ergodic theorem}} + \underbrace{\frac{1}{n}\mathbb{E}(X_{km,n})^+}_{\xrightarrow[n\to\infty]{} 0 \text{ since } |n-km|<m}.$$

This concludes the proof of the subadditive ergodic thm.

Applications

1) Products of random matrices with positive entries (Furstenberg and Kesten 1960).

Let $k \geq 1$ integer. Consider random $k \times k$ matrices $A_1, A_2, \dots$ with a stationary dist. of (A_i) such that all entries of the A_i are strictly positive.

Thm.: (under some moment assumptions) For some RV Y, for each $1 \leq i, j \leq k$,

$$\frac{1}{n}\operatorname{Log}\left[(A_1 \cdot \dots \cdot A_n)(i,j)\right] \to Y \text{ a.s. and in } L_1$$

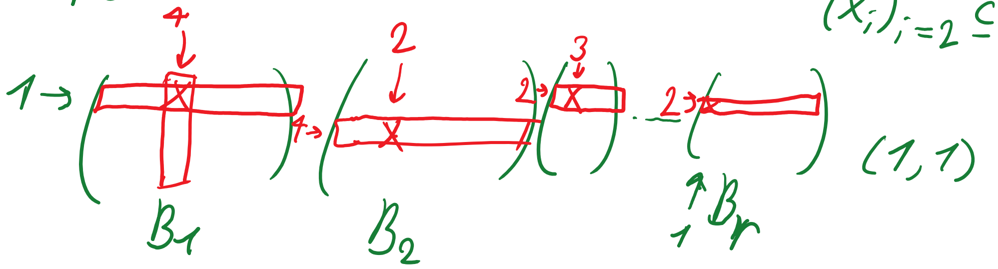
$$\frac{1}{n}\log[(A_1\cdots A_n)(i,j)] \to \gamma \text{ a.s. and in } L_1.$$

Proof: We prove this for $i=j=1$, and the deduction for general i,j is an exercise.

Define $\alpha_{m,n} := \log((A_{m+1}\cdots A_n)(1,1))$.

Then $\alpha_{0,n} \geq \alpha_{0,m} + \alpha_{m,n}$ for all $0<m<n$.

Recall that $B_1\cdots B_r(x_1, x_{r+1}) = \sum_{(x_i)_{i=2}^{r} \subseteq \{1,\dots,k\}} \prod_{i=1}^{r} B_i(x_i, x_{i+1})$



Superadditivity implies subadditivity for $(-\alpha_{m,n})$. Stationarity is clear.

It remains to check the moment bounds (iv).

Once that is checked, the subadditive ergodic thm. shows that $\frac{1}{n}\alpha_{0,n} \to \gamma$ a.s. and in L_1.

Moment bounds: Assume $\mathbb{E}|\log(A_1(i,j))| < \infty$ for all i,j.

Need to show $\mathbb{E}\alpha_{0,1}^- < \infty$ and $\mathbb{E}\alpha_{0,n} \leq \gamma_0 n$ (for $\gamma_0 < \infty$ opposite from the thm. since α is superadditive)

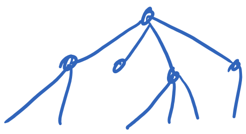
$$\mathbb{E}\alpha_{0,1}^- \leq \mathbb{E}|\log A_1(1,1)| < \infty.$$

$$(A_1\cdots A_n)(1,1) \leq k^{n-1}\prod_{i=1}^{n}\left(\max_{j,m} A_i(j,m)\right)$$

$$\Rightarrow \alpha_{0,n} \leq (n-1)\log k + \sum_{i=1}^{n}\max_{j,m}\log(A_i(j,m)).$$

$$\Rightarrow \frac{1}{n}\mathbb{E}\,\alpha_{0,n} \le \log k + \underbrace{\mathbb{E}\Big(\max_{j,m} \log A_1(j,m)\Big)}_{\le \sum_{j,m} |\log A_1(j,m)|} < \infty$$

2) Age-dependant branching processes (Biggins 1978 proved more)



usual branching process

Age-dependant branching process

Like a Galton-Watson tree, but each individual lives for a certain dist. according to a dist. ν, and has children when it dies.

We further suppose that all individuals have at least one child.

$X_{0,m}$:= birth time of the first member of generation m.

Result: $\frac{1}{n} X_{0,n} \xrightarrow[n\to\infty]{} \gamma$ ← deterministic const. a.s. and in L_1 (under moment assump.)

Idea of proof: Define $X_{m,n}$ to be the time lag for the first individual born at generation m to have a child at gen. n.

Then $X_{0,n} \le X_{0,m} + X_{m,n}$ $\forall\, 0 < m < n$.

(but note that $X_{m,n} \le X_{m,r} + X_{r,n}$)

(but note that $X_{m,n} \leq X_{m,r} + X_{r,n}$
need not hold for $m<r<n$)